A Fluidity Based First-Order System Least-Squares Method for Ice Sheets

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Outline:

1. FOSLS Formulations
2. Test Problems
3. Future Plans
Outline:

1. FOSLS Formulations
   - Viscosity Formulation
   - Fluidity Formulation

2. Test Problems

3. Future Plans
Outline:

1. FOSLS Formulations
   - Viscosity Formulation
   - Fluidity Formulation

2. Test Problems

3. Future Plans
Continuity Equation:

\[ \nabla \cdot u = 0 \]

Momentum Equation:

\[ 0 = \nabla \cdot \mu \left( \nabla u + (\nabla u)^T \right) - \nabla p + \rho g, \]

Viscosity

\[ \mu = \frac{1}{2} \left( \frac{A}{2} \right)^{-\frac{1}{3}} \left\| \dot{\varepsilon} \right\|_{F}^{-\frac{2}{3}}, \]

\[ \dot{\varepsilon} = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right). \]
Rewrite as a First Order System

**Definition**

\[
\mathbf{U} = \nabla \mathbf{u} = \begin{bmatrix}
\frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \frac{\partial u_3}{\partial x} \\
\frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \frac{\partial u_3}{\partial y} \\
\frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} & \frac{\partial u_3}{\partial z}
\end{bmatrix} = \begin{bmatrix}
U_{11} & U_{21} & U_{31} \\
U_{12} & U_{22} & U_{32} \\
U_{13} & U_{23} & U_{33}
\end{bmatrix}
\]

**First Order System:**

\[
\nabla \cdot \mathbf{u} = 0 \quad \text{(Continuity)}
\]

\[
\mathbf{U} = \nabla \mathbf{u} \quad \text{(Definition)}
\]

\[
\nabla \cdot \frac{1}{2} \mu (\mathbf{U} + \mathbf{U}^T) - \nabla p = -\rho g \quad \text{(Momentum)}
\]
Rewrite as a First Order System

**Definition**

\[
\mathbf{U} = \nabla \mathbf{u} = \begin{bmatrix}
\frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \frac{\partial u_3}{\partial x} \\
\frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \frac{\partial u_3}{\partial y} \\
\frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} & \frac{\partial u_3}{\partial z}
\end{bmatrix} = \begin{bmatrix}
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**First Order System:**

\[
\nabla \cdot \mathbf{u} = 0 \quad \text{(Continuity)}
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\mathbf{U} = \nabla \mathbf{u} \quad \text{(Definition)}
\]

\[
\nabla \cdot \frac{1}{2} \mu (\mathbf{U} + \mathbf{U}^T) - \nabla p = -\rho g \quad \text{(Momentum)}
\]

\[
\nabla \times \mathbf{U} = 0 \quad \text{(Curl of Definition)}
\]
Rewrite as a First Order System

**Definition**

\[
\underline{U} = \nabla \underline{u} = \begin{bmatrix}
\frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \frac{\partial u_3}{\partial x} \\
\frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \frac{\partial u_3}{\partial y} \\
\frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} & \frac{\partial u_3}{\partial z}
\end{bmatrix} = \begin{bmatrix}
U_{11} & U_{21} & U_{31} \\
U_{12} & U_{22} & U_{32} \\
U_{13} & U_{23} & U_{33}
\end{bmatrix}
\]

**First Order System:**

\[
\nabla \cdot \underline{u} = 0 \quad \text{(Continuity)}
\]
\[
\underline{U} = \nabla \underline{u} \quad \text{(Definition)}
\]
\[
\nabla \cdot \frac{1}{2} \mu (\underline{U} + \underline{U}^T) - \nabla p = -\rho g \quad \text{(Momentum)}
\]
\[
\nabla \times \underline{U} = 0 \quad \text{(Curl of Definition)}
\]
\[
\text{Trace}(\underline{U}) = 0
\]
Rewrite as a First Order System

**Definition**

\[ \underline{U} = \nabla \underline{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \frac{\partial u_3}{\partial x} \\ \frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \frac{\partial u_3}{\partial y} \\ \frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} & \frac{\partial u_3}{\partial z} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{21} & U_{31} \\ U_{12} & U_{22} & U_{32} \\ U_{13} & U_{23} & U_{33} \end{bmatrix} \]

**Viscosity Formulation:**

\[ \nabla \cdot \underline{u} = 0 \quad \text{(Continuity)} \]

\[ \underline{U} = \nabla \underline{u} \quad \text{(Definition)} \]

\[ \nabla \cdot \frac{1}{2} \mu (\underline{U} + \underline{U}^T) - \nabla p = -\rho g \quad \text{(Momentum)} \]

\[ \nabla \times \underline{U} = 0 \quad \text{(Curl of Definition)} \]

\[ \text{Trace}(\underline{U}) = 0 \quad \text{(Enforced by setting} \ U_{11} = -U_{22}) \]
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A problem with this formulation comes in the definition for viscosity.

\[ \mu = \frac{1}{2} A^{-\frac{1}{3}} \dot{\varepsilon}_e^{-\frac{2}{3}} \]

\[ \dot{\varepsilon}_e = \frac{1}{\sqrt{2}} \left\| \ddot{\varepsilon} \right\|_F \]

\[ \ddot{\varepsilon} = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right) \]

The viscosity is near infinite where the glacier experiences small deformations. This is usually overcome by using a small constant in the effective strain rate.
Fluidity Formulation
Move Constants

Viscosity

\[ \mu = c_A \left\| \dot{\varepsilon} \right\|_F^{-\frac{2}{3}}, \]
\[ c_A = \frac{1}{2} \left( \frac{A}{2} \right)^{-\frac{1}{3}}, \]

Momentum Equation:

\[ 0 = \nabla \cdot \hat{\mu} \left( \nabla u + (\nabla u)^T \right) - \nabla \hat{p} + \hat{\rho} g, \]
\[ \hat{\mu} = \left\| \dot{\varepsilon} \right\|_F^{-\frac{2}{3}}, \quad \hat{p} = \frac{p}{c_A}, \quad \hat{\rho} = \frac{\rho}{c_A}. \]
Another way to turn Stokes for glaciers into a first order system is to start with the deviatoric stress.

\[
\sigma' = 2\dot\mu \ddot\varepsilon = \dot\mu (\nabla u + (\nabla u)^T)
\]

\[
\begin{bmatrix}
\sigma'_{11} & \sigma'_{12} \\
\sigma'_{12} & \sigma'_{22}
\end{bmatrix} = \begin{bmatrix}
2\dot\mu \partial_x u_1 & \dot\mu (\partial_y u_1 + \partial_x u_2) \\
\dot\mu (\partial_y u_1 + \partial_x u_2) & 2\dot\mu \partial_y u_2
\end{bmatrix}
\]
Another way to turn Stokes for glaciers into a first order system is to start with the deviatoric stress.

\[ \sigma' = 2\hat{\mu}\ddot{\varepsilon} = \hat{\mu} (\nabla u + (\nabla u)^T) \]

\[
\begin{bmatrix}
\sigma'_{11} & \sigma'_{12} \\
\sigma'_{12} & \sigma'_{22}
\end{bmatrix} =
\begin{bmatrix}
2\hat{\mu}\partial_x u_1 & \hat{\mu}(\partial_y u_1 + \partial_x u_2) \\
\hat{\mu}(\partial_y u_1 + \partial_x u_2) & 2\hat{\mu}\partial_y u_2
\end{bmatrix}
\]
Another way to turn Stokes for glaciers into a first order system is to start with the deviatoric stress.

\[
\sigma' = 2\hat{\mu}\hat{\varepsilon} \\
= \hat{\mu} (\nabla u + (\nabla u)^T)
\]

\[
\begin{bmatrix}
\sigma'_{11} & \sigma'_{12} \\
\sigma'_{12} & \sigma'_{22}
\end{bmatrix} = 
\begin{bmatrix}
2\hat{\mu}\partial_x u_1 & \hat{\mu}(\partial_y u_1 + \partial_x u_2) \\
\hat{\mu}(\partial_y u_1 + \partial_x u_2) & 2\hat{\mu}\partial_y u_2
\end{bmatrix}
\]
Another way to turn Stokes for glaciers into a first order system is to start with the deviatoric stress.

\[
\sigma' = 2\hat{\mu}\dot{\varepsilon}
\]

\[
= \hat{\mu} \left( \nabla u + (\nabla u)^T \right)
\]

\[
\begin{bmatrix}
\sigma'_{11} & \sigma'_{12} \\
\sigma'_{12} & -\sigma'_{11}
\end{bmatrix}
= \begin{bmatrix}
2\hat{\mu}\partial_x u_1 & \hat{\mu}(\partial_y u_1 + \partial_x u_2) \\
\hat{\mu}(\partial_y u_1 + \partial_x u_2) & 2\hat{\mu}\partial_y u_2
\end{bmatrix}
\]

Using the Continuity Equation, \(2\partial_x u_1\) can be split into \(\partial_x u_1 - \partial_y u_2\)
Another way to turn Stokes for glaciers into a first order system is to start with the deviatoric stress.

\[
\begin{align*}
\sigma' &= 2\hat{\mu} \dot{\varepsilon} \\
&= \hat{\mu} \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \\
\begin{bmatrix}
\sigma'_{11} & \sigma'_{12} \\
\sigma'_{12} & -\sigma'_{11}
\end{bmatrix} &= \begin{bmatrix}
\hat{\mu} (\partial_x u_1 - \partial_y u_2) & \hat{\mu} (\partial_y u_1 + \partial_x u_2) \\
\hat{\mu} (\partial_y u_1 + \partial_x u_2) & \hat{\mu} (\partial_y u_2 - \partial_x u_1)
\end{bmatrix}
\end{align*}
\]
Another way to turn Stokes for glaciers into a first order system is to start with the deviatoric stress.

\[
\sigma' = 2\hat{\mu}\ddot{\varepsilon} = \hat{\mu} \left( \nabla u + (\nabla u)^T \right)
\]

\[
\begin{bmatrix}
\hat{\mu}^{-1}\sigma'_{11} & \hat{\mu}^{-1}\sigma'_{12} \\
\hat{\mu}^{-1}\sigma'_{12} & -\hat{\mu}^{-1}\sigma'_{11}
\end{bmatrix} = \begin{bmatrix}
\partial_x u_1 - \partial_y u_2 & \partial_y u_1 + \partial_x u_2 \\
\partial_y u_1 + \partial_x u_2 & \partial_y u_2 - \partial_x u_1
\end{bmatrix}
\]

define the fluidity as \( \phi = \hat{\mu}^{-1} \)
Another way to turn Stokes for glaciers into a first order system is to start with the deviatoric stress.

\[
\sigma' = 2\hat{\mu}\varepsilon \\
= \hat{\mu} \left( \nabla u + (\nabla u)^T \right)
\]

\[
\begin{bmatrix}
\phi\sigma'_{11} & \phi\sigma'_{12} \\
\phi\sigma'_{12} & -\phi\sigma'_{11}
\end{bmatrix} = 
\begin{bmatrix}
\partial_x u_1 - \partial_y u_2 & \partial_y u_1 + \partial_x u_2 \\
\partial_y u_1 + \partial_x u_2 & \partial_y u_2 - \partial_x u_1
\end{bmatrix}
\]

Now we have the two equations:

\[
\phi\sigma'_{11} - (\partial_x u_1 - \partial_y u_2) = 0
\]

\[
\phi\sigma'_{12} - (\partial_y u_1 + \partial_x u_2) = 0
\]
Now we just need to figure out what $\phi$ is in terms of $\sigma'_{11}$ and $\sigma'_{12}$.

\[
\hat{\mu} = \left| \left| \dot{\varepsilon} \right| \right|_F^2 \left| \left| \dot{\varepsilon} \right| \right|_F^{-2} \\
= \hat{\mu}^2 \left| \left| \dot{\varepsilon} \right| \right|_F^2 \\
= \left| \left| \hat{\mu} \dot{\varepsilon} \right| \right|_F^2 \\
= \left| \left| \sigma' / 2 \right| \right|_F^2 \\
= \frac{1}{2} (\sigma'_{11}^2 + \sigma'_{12}^2)
\]
Now we just need to figure out what $\phi$ is in terms of $\sigma'_{11}$ and $\sigma'_{12}$.

$$\hat{\mu}^{-1} = \left\| \dot{\varepsilon} \right\|_{F}^{2}$$

$$= \left\| \dot{\varepsilon} \right\|_{F}^{2} \frac{4}{3} \left\| \dot{\varepsilon} \right\|_{F}^{2}$$

$$= \hat{\mu}^{2} \left\| \dot{\varepsilon} \right\|_{F}^{2}$$

$$= \left\| \hat{\mu} \dot{\varepsilon} \right\|_{F}^{2}$$

$$= \left\| \sigma'/2 \right\|_{F}^{2}$$

$$= \frac{1}{2} \left( \sigma'_{11}^{2} + \sigma'_{12}^{2} \right)$$
Now we just need to figure out what $\phi$ is in terms of $\sigma'_{11}$ and $\sigma'_{12}$.

$$
\hat{\mu}^{-1} = \left\| \dot{\varepsilon} \right\|_F \left( \varepsilon \right)^{\frac{2}{3}} \\
= \left\| \dot{\varepsilon} \right\|_F \left( \varepsilon \right)^{\frac{2}{3}} \\
= \hat{\mu}^2 \left\| \dot{\varepsilon} \right\|_F \\
= \left\| \hat{\mu} \dot{\varepsilon} \right\|_F \\
= \left\| \sigma' / 2 \right\|_F \\
= \frac{1}{2} (\sigma'_{11}^2 + \sigma'_{12}^2)
$$
Now we just need to figure out what $\phi$ is in terms of $\sigma'_{11}$ and $\sigma'_{12}$.

\[
\hat{\mu}^{-1} = \left| \hat{\varepsilon} \right|_{F}^{3/2} \\
= \left| \hat{\varepsilon} \right|_{F}^{-4/3} \left| \hat{\varepsilon} \right|_{F}^{2} \\
= \hat{\mu}^{2} \left| \hat{\varepsilon} \right|_{F}^{2} \\
= \left| \hat{\mu} \dot{\varepsilon} \right|_{F}^{2} \\
= \left| \sigma'_{/2} \right|_{F}^{2} \\
= \frac{1}{2} \left( \sigma'_{11}^{2} + \sigma'_{12}^{2} \right)
\]
Now we just need to figure out what $\phi$ is in terms of $\sigma'_{11}$ and $\sigma'_{12}$.

\[
\hat{\mu}^{-1} = \frac{2}{3} \frac{\|\dot{\varepsilon}\|}{F} \\
= \frac{4}{3} \frac{\|\dot{\varepsilon}\|_F^3 \|\dot{\varepsilon}\|_F^2}{F} \\
= \hat{\mu}^2 \frac{\|\dot{\varepsilon}\|_F^2}{F} \\
= \frac{\|\hat{\mu} \dot{\varepsilon}\|_F^2}{F} \\
= \frac{\|\sigma'/2\|_F^2}{F} \\
= \frac{1}{2} (\sigma'_{11}^2 + \sigma'_{12}^2)
\]
Now we just need to figure out what $\phi$ is in terms of $\sigma'_{11}$ and $\sigma'_{12}$.

\[
\hat{\mu}^{-1} = \left| \| \dot{\varepsilon} \|_F^3 \right|^\frac{2}{3} \\
= \left| \| \dot{\varepsilon} \|_F^4 \right| \frac{1}{3} \left| \| \dot{\varepsilon} \|_F^2 \right|_F \\
= \hat{\mu}^2 \left| \| \dot{\varepsilon} \|_F^2 \right|_F \\
= \left| \| \hat{\mu} \dot{\varepsilon} \|_F^2 \right|_F \\
= \left| \| \sigma'/2 \|_F^2 \right| \\
= \frac{1}{2} \left( \sigma'_{11}^2 + \sigma'_{12}^2 \right)
\]
Now we just need to figure out what \( \phi \) is in terms of \( \sigma'_{11} \) and \( \sigma'_{12} \).

\[
\hat{\mu}^{-1} = \left| \dot{\varepsilon} \right|^{2} \frac{2}{F}
\]

\[
= \left| \dot{\varepsilon} \right|^{\frac{4}{3}} \frac{2}{F}
\]

\[
= \hat{\mu}^{2} \left| \dot{\varepsilon} \right|^{\frac{2}{F}}
\]

\[
= \left| \hat{\mu} \dot{\varepsilon} \right|^{\frac{2}{F}}
\]

\[
= \left| \sigma'_{ij} / 2 \right|^{\frac{2}{F}}
\]

\[
= \frac{1}{2} \left( \sigma'_{11}^{2} + \sigma'_{12}^{2} \right)
\]
Now we just need to figure out what $\phi$ is in terms of $\sigma_{11}'$ and $\sigma_{12}'$.

$$
\hat{\mu}^{-1} = |\hat{\dot{\varepsilon}}| \frac{2}{3} F \\
= |\hat{\dot{\varepsilon}}| \left( \frac{4}{3} F \right)^{\frac{2}{3} |\hat{\dot{\varepsilon}}|} F \\\n= \hat{\mu}^2 |\hat{\dot{\varepsilon}}| F \\\n= |\hat{\mu} \hat{\dot{\varepsilon}}| F \\\n= |\tilde{\sigma}' / 2| F \\\n\phi = \frac{1}{2} (\sigma_{11}'^2 + \sigma_{12}'^2)
$$
Fluidity Formulation

Stress-Gradient-Fluidity

Stress-Gradient-Fluidity Formulation

\[ \phi \sigma'_{11} - (\partial_x u_1 - \partial_y u_2) = 0, \]
\[ \phi \sigma'_{12} - (\partial_y u_1 + \partial_x u_2) = 0, \]
\[ \partial_x u_1 + \partial_y u_2 = 0, \]

Definition of \( \sigma' \)

Continuity Equation

\[ -\partial_x \sigma'_{11} - \partial_y \sigma'_{12} + \partial_x \hat{p} - f_1 = 0, \]
\[ \partial_y \sigma'_{11} - \partial_x \sigma'_{12} + \partial_y \hat{p} - f_2 = 0, \]

Momentum Equation

where \( \underline{f} = [f_1, f_2] = \hat{\rho} g \).

Jeffery Allen
Glacial FOSLS
February 10, 2016
Fluidity Formulation

Vorticity

Just like in the Viscosity formulation was want to add an analogous curl equation. First we must define vorticity:

$$\psi \omega - (-\partial_y u_1 + \partial_x u_2) = 0.$$ 

To match the form of $\phi$ we let

$$\psi = \frac{1}{2}(\sigma'_{11}^2 + \sigma'_{12}^2 + \omega^2).$$

Using the equations for vorticity, continuity, and the definition of $\sigma'$, we can construct the gradient of $u$.

$$\nabla u = \begin{bmatrix} \partial_x u_1 & \partial_y u_1 \\ \partial_x u_2 & \partial_y u_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \phi \sigma'_{11} & \phi \sigma'_{12} - \psi \omega \\ \phi \sigma'_{12} + \psi \omega & -\phi \sigma'_{11} \end{bmatrix}$$
We can now add $\nabla \times \nabla \vec{u} = 0$ to our system to get:

**Stress-Gradient-Vorticity-Fluidity Formulation**

\[
\begin{align*}
\phi \sigma'_{11} - (\partial_x u_1 - \partial_y u_2) &= 0, \\
\phi \sigma'_{12} - (\partial_y u_1 + \partial_x u_2) &= 0, \\
\psi \omega - (-\partial_y u_1 + \partial_x u_2) &= 0, \\
\end{align*}
\]

Definition of $\sigma'$, $\omega$

\[
\begin{align*}
\partial_x u_1 + \partial_y u_2 &= 0, \\
-\partial_x \sigma'_{11} - \partial_y \sigma'_{12} + \partial_x \hat{p} - f_1 &= 0, \\
\partial_y \sigma'_{11} - \partial_x \sigma'_{12} + \partial_y \hat{p} - f_2 &= 0, \\
-\partial_y (\phi \sigma'_{11}) + \partial_x (\phi \sigma'_{12}) - \partial_x (\psi \omega) &= 0, \\
-\partial_x (\phi \sigma'_{11}) - \partial_y (\phi \sigma'_{12}) - \partial_y (\psi \omega) &= 0.
\end{align*}
\]

Continuity Equation

Momentum Equation

Curl Equation
Outline:

1. FOSLS Formulations

2. Test Problems
   - Rectangular Domain
   - Rec Domain - Results
   - ISMIP-HOM Benchmark B
   - BenchB - Results

3. Future Plans
Outline:

1. FOSLS Formulations

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3. Future Plans
with \( \mathbf{g} = |g|[\sin(\theta), \cos(\theta)]^T \), \( \hat{H} = 1 \) km, and \( \hat{L} = 10 \) km.

<table>
<thead>
<tr>
<th>Bottom</th>
<th>Top</th>
<th>Ends</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_1 = 0 )</td>
<td>( \mathbf{\sigma}' - \hat{p} \mathbf{I} ) \cdot \mathbf{n} = 0 )</td>
<td>Periodic</td>
</tr>
<tr>
<td>( u_2 = 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \nabla \mathbf{u} \cdot \mathbf{t} = 0 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
For the Top boundary we want to impose a stress free condition \((\overline{\sigma} \cdot n = 0)\).

\[
\overline{\sigma} \cdot n = (\overline{\sigma'} - \hat{p} I) \cdot n
\]

\[
= \begin{bmatrix}
\sigma_{11}' - \hat{p} & \sigma_{12}' \\
\sigma_{12}' & -\sigma_{12}' - \hat{p}
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\sigma_{12}' \\
\sigma_{11}' + \hat{p}
\end{bmatrix} = 0.
\]
Assume the glacier is frozen to the bed (no slip)

\[ u = 0 \]

This also gives us:

\[ \nabla u \cdot t = 0 \]
Assume the glacier is frozen to the bed (no slip)

$$u = 0$$

This also gives us:

$$\nabla u \cdot \mathbf{t} = \begin{bmatrix} \frac{\partial x u_1}{\partial x} & \frac{\partial y u_1}{\partial y} \\ \frac{\partial x u_2}{\partial x} & \frac{\partial y u_2}{\partial y} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$
Assume the glacier is frozen to the bed (no slip)

\[ u = 0 \]

This also gives us:

\[
\nabla \underline{u} \cdot t = \frac{1}{2} \begin{bmatrix} \phi \sigma'_{11} & \phi \sigma'_{12} - \psi \omega \\ \phi \sigma'_{12} + \psi \omega & -\phi \sigma'_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0
\]
Assume the glacier is frozen to the bed (no slip)

\[ u = 0 \]

This also gives us:

\[ \nabla u \cdot t = \left[ \phi \sigma_{11} + \psi \omega \right] = 0 \]
Assume the glacier is frozen to the bed (no slip)

\[ u = 0 \]

This also gives us:

\[ \nabla u \cdot t = \left[ \begin{array}{c} \frac{\phi \sigma_{11}'}{\psi} \\ \phi \sigma_{12}' + \omega \end{array} \right] = 0 \]
Assume the glacier is frozen to the bed (no slip)

\[ u = 0 \]

This also gives us:

\[
\nabla u \cdot t = \begin{bmatrix} \sigma'_{11} \\ \phi \sigma'_{12} + \omega \end{bmatrix} = 0
\]

Finally, assume periodic side boundaries.
Rectangular Glacier

Exact Solution

Downhill Velocity Profile

Problematic Part of Viscosity
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3. Future Plans
Rectangular Glacier
Least Squares Functional

Functional Convergence

- Fluidity
- Viscosity

$O(h)$
$O(h^2)$

Degrees of Freedom

$\|L u^h - f\|$
$L^2$ Convergence

- Fluidity
- Viscosity
- $O(h^2)$
- $O(h^3)$

Degrees Of Freedom

L2 Error

1.0
0.1
0.01
1.0e-4
1.0e-6
1.0e-8

1200 4320 16320 63360 249600 990720 3947520

Jeffery Allen
Glacial FOSLS
February 10, 2016
## Rectangular Glacier
### Work Units

<table>
<thead>
<tr>
<th>Level</th>
<th>E</th>
<th>Nonzeros</th>
<th>N</th>
<th>Comp</th>
<th>V-Cycles</th>
<th>WU</th>
<th>Functional</th>
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<tbody>
<tr>
<td>Viscosity Formulation</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>80</td>
<td>72000</td>
<td>22</td>
<td>1.57</td>
<td>60.82</td>
<td>1.49</td>
<td>8.52 × 10⁻²</td>
</tr>
<tr>
<td>2</td>
<td>320</td>
<td>276480</td>
<td>4</td>
<td>1.78</td>
<td>55.25</td>
<td>1.07</td>
<td>2.85 × 10⁻²</td>
</tr>
<tr>
<td>3</td>
<td>1280</td>
<td>1082880</td>
<td>6</td>
<td>1.94</td>
<td>96.67</td>
<td>11.94</td>
<td>1.05 × 10⁻²</td>
</tr>
<tr>
<td>4</td>
<td>5120</td>
<td>4285440</td>
<td>10</td>
<td>1.97</td>
<td>112.90</td>
<td>93.57</td>
<td>2.40 × 10⁻³</td>
</tr>
<tr>
<td>5</td>
<td>20480</td>
<td>17049600</td>
<td>8</td>
<td>2.01</td>
<td>129.00</td>
<td>346.70</td>
<td>5.98 × 10⁻⁴</td>
</tr>
<tr>
<td>6</td>
<td>81920</td>
<td>68014080</td>
<td>6</td>
<td>2.03</td>
<td>130.30</td>
<td>1061.00</td>
<td>1.49 × 10⁻⁴</td>
</tr>
<tr>
<td>7</td>
<td>327680</td>
<td>271687680</td>
<td>3</td>
<td>2.04</td>
<td>124.30</td>
<td>2033.00</td>
<td>3.76 × 10⁻⁵</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>Total 3550.00</td>
</tr>
</tbody>
</table>

| Fluidity Formulation |     |          |    |      |          |    |            |
| 1     | 80  | 72000    | 4  | 1.34 | 13.50    | 0.05| 1.36 × 10⁻¹ |
| 2     | 320 | 276480   | 3  | 1.62 | 14.67    | 0.19| 3.25 × 10⁻² |
| 3     | 1280| 1082880  | 3  | 1.78 | 16.67    | 0.94| 8.21 × 10⁻³ |
| 4     | 5120| 4285440  | 3  | 1.86 | 17.67    | 4.15| 2.06 × 10⁻³ |
| 5     | 20480| 17049600 | 2  | 1.91 | 18.00    | 11.49| 5.15 × 10⁻⁴ |
| 6     | 81920| 68014080 | 1  | 1.95 | 17.00    | 22.11| 1.29 × 10⁻⁴ |
| 7     | 327680| 271687680| 1  | 1.96 | 19.00    | 99.34| 3.22 × 10⁻⁵ |
|       |     |          |    |      |          |    | Total 138.30 |
Outline:

1. FOSLS Formulations

2. Test Problems
   - Rectangular Domain
   - Rec Domain - Results
   - ISMIP-HOM Benchmark B
   - BenchB - Results

3. Future Plans
The surface of the glacier is prescribed by

\[ z_s(x) = -\tan(\theta)x, \]

and the basal topography is prescribed by

\[ z_b(x) = z_s(x) - H + \beta H \sin(wx). \]
Benchmark B
Boundary Conditions

\[
\begin{align*}
\text{Bottom} & \quad \text{Top} & \quad \text{Ends} \\
\begin{aligned}
    u_1 &= 0 \\
    u_2 &= 0 \\
    \nabla u \cdot t &= 0
\end{aligned} & \quad \begin{aligned}
    (\sigma' - \hat{p}I) \cdot n &= 0 \\
\end{aligned} & \quad \text{Periodic}
\end{align*}
\]
Outline:

1. FOSLS Formulations

2. Test Problems
   - Rectangular Domain
   - Rec Domain - Results
   - ISMIP-HOM Benchmark B
   - BenchB - Results

3. Future Plans
Horizontal Surface Velocity: $u_1(x, z_5(x))$
Vertical Surface Velocity: $u_2(x, z_s(x))$
Basal Shear Stress: $\sigma_{xz}(x, z_b(x))$
Benchmark B

Benchmark Plots

Pressure Deviation from Hydrostatic

Distance (km)

Pressure (kPa)

-40
-20
0
20
40
0 2 4 6 8 10

-Fluidity
- Taylor–Hood
- Stokes
- HOM
- Mean

Jeffery Allen
Glacial FOSLS
February 10, 2016 20 / 22
Outline:

1. FOSLS Formulations
2. Test Problems
3. Future Plans
Future Work

1. Mass Conservation Study
2. Other Flow Laws
3. More Efficient Iterative Solver ($H(div)$ or $H^1$)
4. ISMIP-HOM: Benchmark D (basal sliding)
5. Ice Shelf Modeling/Grounding Line Determination
6. Time Dependant Domains?
Questions?

Questions?

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